Indag. Mathem., N.S., 16 (2), 237–250 June 20, 2005

Approximate extension of partial ε-characters of abelian groups to characters with application to integral point lattices ☆

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Communicated by Prof. J.J. Duistermaat at the meeting of December 20, 2004

ABSTRACT

Let G be an abelian group, $S \subseteq G$ be a finite set, and $\mathbb{T}$ denote the multiplicative group of complex units with the invariant arc metric $|\arg(a/b)|$. We will show that for a mapping $f : S \to \mathbb{T}$ to be ε-close on S to a character $\varphi : G \to \mathbb{T}$ it is enough that f be extendable to a mapping $\bar{f} : (S \cup \{1\} \cup S^{-1})^n \to \mathbb{T}$, where n is big enough and $\bar{f}$ violates the homomorphy condition at most up to an arbitrary $\delta < \min(\varepsilon, \pi/2)$. Moreover, n can be chosen uniformly, independently of G and both f and $\bar{f}$, depending just on δ, ε and the number of elements of S. The proof is non-constructive, using the ultraproduct construction and Pontryagin duality, hence yielding no estimate on the actual size of n. As one of the applications we show that, for a vector $u \in \mathbb{R}^q$ to be ε-close to some vector from the dual lattice $H^\star$ of a full rank integral point lattice $H \subseteq \mathbb{Z}^q$, it is enough for the scalar product ux to be δ-close (with $\delta < 1/3$) to an integer for all vectors $x \in H$ satisfying $\sum_i |x_i| \leqslant n$, where n depends on δ, ε and q only.

1. INTRODUCTION

Let G, H be groups, the latter endowed with a (left) invariant metric ρ, and $\varepsilon > 0$. A mapping $f : S \to H$, where $S \subseteq G$, is called a *partial ε-homomorphism* if $\rho(f(xy), f(x)f(y)) \leqslant \varepsilon$ for all $x, y \in S$ such that $xy \in S$. If $S = G$ then f is called an *ε-homomorphism*. If $f : S \to H$ satisfies the homomorphy condition $f(xy) = f(x)f(y)$ whenever $x, y, xy \in S$, then f is called a *partial homomorphism*.

MSC: primary 20K30; secondary 11H06, 11U07

Key words and phrases: Abelian group, Dual group, Partial ε-homomorphism, Approximate extension, Ulam's problem, Integral point lattice, Dual lattice

☆ Research supported by the Grant No. 1/0261/03 of the Slovak Scientific Grant Agency VEGA.
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Two mappings $f_1 : S_1 \to H$, $f_2 : S_2 \to H$, where $S_1, S_2 \subseteq G$, are said to be *ε-close on a set $S \subseteq S_1 \cap S_2$* if $\rho(f_1(x), f_2(x)) \leqslant \varepsilon$ for each $x \in S$.

Some conditions under which a (continuous) δ-homomorphism $f : G \to H$ is ε-close on G to a (continuous) homomorphism $\varphi : G \to H$ were studied for example in [11,7,8,2,16]; cf. also the extensive reference lists in [10] and [18]. In the present paper we will examine the problem when a partial δ-homomorphism $f : R \to \mathbb{T}$ from a finite subset R of an abelian group G to the multiplicative group of all complex units $\mathbb{T}$ with the angular metric $|\arg(a/b)|$ is ε-close to a homomorphism $\varphi : G \to \mathbb{T}$ on a set $S \subseteq R$. Alternatively, we will use terms like *(partial) ε-character* and *(partial) character*.

Not even all partial homomorphisms can be extended to homomorphisms. The necessary and sufficient condition can easily be stated: A partial homomorphism $f : S \to H$, defined on a subset S of a group G extends to a homomorphism $\varphi : \langle S \rangle \to H$ if and only if, for any integer $n > 0$ and all $x_1, \ldots, x_n \in S$, the equality $x_1 \cdots x_n = 1$ in G implies the equality $f(x_1) \cdots f(x_n) = 1$ in H, or, equivalently, if f extends to a partial homomorphism $\langle S \rangle_n \to H$ for each $n > 0$, where $\langle S \rangle_n = (S \cup \{1\} \cup S^{-1})^n$ and $\langle S \rangle = \bigcup_{n \in \mathbb{N}} \langle S \rangle_n$ is the subgroup of G generated by S. For G abelian and $H = \mathbb{T}$, this automatically implies the extendability of f to a character $\varphi : G \to \mathbb{T}$.

As a finite set $S \subseteq G$ may contain elements of arbitrarily big order, there seems to be no reason for the existence of an integer n, depending just on the number $\#S$ of elements of S, such that the extendability of $f : S \to \mathbb{T}$ to a partial character $\langle S \rangle_n \to \mathbb{T}$ would guarantee its extendability to a character $\varphi : G \to \mathbb{T}$ for *all* G abelian, S and f. Therefore it is perhaps surprising that the approximative version of this result is true: If $0 < \delta < \varepsilon \leqslant \pi/2$, $f : \langle S \rangle_n \to \mathbb{T}$ is a δ-character and n, depending just on δ, ε and $\#S$, is big enough, then f is already ε-close on S to a character $\varphi : G \to \mathbb{T}$. The proof, in spite of fairly elementary formulation of the theorem, uses relatively heavy machinery of ultraproducts and the Pontryagin–van Kampen duality. As a consequence, it is non-constructive and yields no actual estimate for n.

In the last paragraph we present some applications to dual lattices of integral point lattices. One typical result (Corollary 5.5) reads as follows: If $0 < \delta < 1/3$, $\varepsilon > 0$ and $1 \leqslant q \in \mathbb{N}$, then there exists an $n \in \mathbb{N}$, depending just on δ, ε and q, such that for every full rank integral point lattice $H \subseteq \mathbb{Z}^q$ the following holds: if $u \in \mathbb{R}^q$ is a vector such that for all $x \in H$ satisfying $\sum_{i=1}^{q} |x_i| \leqslant n$ the scalar product ux is δ-close to an integer, then u is already ε-close to some vector from the dual lattice $H^\star$. Again, it is not clear how big the n actually has to be.

We conclude with formulation of some open problems concerning the provability status of the above mentioned results within a more elementary framework.

We start with two paragraphs of preparatory character restating some results by Kazhdan [11] and Gordon [5,6] in a form better suited for our purpose.

2. KAZHDAN'S THEOREM

Recall that an *amenable group* G is a locally compact group, endowed with an *invariant mean* M, see, e.g., [14]. I.e., $M : L_\infty(G) \to \mathbb{C}$ is a (left) invariant positive linear functional, assigning the value 1 to the constant function $1 : G \to \mathbb{C}$.

As shown in [11], if G is an amenable group, $H = U(X)$ is the group of all unitary operators on some Hilbert space X with the usual operator norm, and $\varepsilon < 1/200$, then any (continuous) ε-homomorphism $f: G \to H$ is 2ε-close to a (continuous) homomorphism $\varphi: G \to H$. A more elementary proof, working for amenable G and finite dimensional compact Lie group H, can be found in [2]. In case $H = \mathbb{T} = U(\mathbb{C})$ which is sufficient for our purpose, we give even a more elementary proof, under a considerably weaker restriction on ε and a better estimation of the distance of both maps. For convenience sake, we will consider the *arc* or *angular metric* $|\arg(a/b)|$ on $\mathbb{T}$, defined by means of the argument function arg which is a diffeomorphism of $\mathbb{T} \setminus \{-1\}$ onto the open interval $(-\pi, \pi)$, and $\arg(-1) = \pi$. The switch between the arc metric and the euclidean metric $|a - b|$, inherited from the complex plane, is provided by the formula

$$|a - b| = 2 \sin \frac{1}{2} \left| \arg \frac{a}{b} \right|.$$

Thus the results of Theorems 2.1 and 4.1 below can readily be restated for the latter, just changing the bound for ε from $\pi/2$ to $\sqrt{2}$.

Theorem 2.1. *Let G be an amenable locally compact group. If $0 < \varepsilon < \pi/2$ then for every ε-homomorphism $f: G \to \mathbb{T}$ there exists a homomorphism $\varphi: G \to \mathbb{T}$ such that*

$$\left| \arg \frac{\varphi(x)}{f(x)} \right| \leqslant \varepsilon$$

for each $x \in G$. Moreover, if f is continuous then one can assume the same for φ.

Proof. Let $0 < \varepsilon < \pi/2$, and $f: G \to \mathbb{T}$ be an ε-homomorphism. Define $\varphi: G \to \mathbb{T}$ by

$$\varphi(x) = f(x) \exp\left(\mathrm{i} M_t \left[\arg \frac{f(xt)}{f(x) f(t)} \right] \right),$$

where M_t denotes the invariant mean M on G with the argument regarded as a function of t. Thus we have

$$\frac{\varphi(x)}{f(x)} = \exp\left(\mathrm{i} M_t \left[\arg \frac{f(xt)}{f(x) f(t)} \right] \right),$$

as well as

$$\left| \arg \frac{\varphi(x)}{f(x)} \right| \leqslant \varepsilon,$$

for any $x \in G$. Moreover, if f is continuous then, obviously, so is φ. Thus it remains to show that φ is a group homomorphism.

Expressing the quotient $f(xyt)/f(x)f(y)f(t)$ in two ways, we obtain

$$\frac{f(xyt)}{f(xy)f(t)}\,\frac{f(xy)}{f(x)f(y)}=\frac{f(xyt)}{f(x)f(yt)}\,\frac{f(yt)}{f(y)f(t)}$$

for any $x, y, t \in G$. As each of the above four quotients is in absolute value $\leqslant \varepsilon < \pi/2$, the argument of the product on each side equals the sum of the arguments of the corresponding factors, yielding

$$\arg\frac{f(xyt)}{f(xy)f(t)}+\arg\frac{f(xy)}{f(x)f(y)}=\arg\frac{f(xyt)}{f(x)f(yt)}+\arg\frac{f(yt)}{f(y)f(t)}.$$

Applying the mean M_t to both sides, and using the substitution $yt = s$ and the left invariance of M in the first right-hand side summand, we get

$$\begin{aligned}M_t&\left[\arg\frac{f(xyt)}{f(xy)f(t)}\right]+M_t\left[\arg\frac{f(xy)}{f(x)f(y)}\right]\\&=M_s\left[\arg\frac{f(xs)}{f(x)f(s)}\right]+M_t\left[\arg\frac{f(yt)}{f(y)f(t)}\right].\end{aligned}$$

Taking the exponential of i times each side and realizing that the second left side term is independent of t, as well as the identities $\exp\mathrm{i}(u+v)=\exp\mathrm{i}u\exp\mathrm{i}v$, for $u, v \in \mathbb{R}$, and $\exp(\mathrm{i}\arg z) = z$, for $z \in \mathbb{T}$, we obtain

$$\frac{\varphi(xy)}{f(xy)}\,\frac{f(xy)}{f(x)f(y)}=\frac{\varphi(x)}{f(x)}\,\frac{\varphi(y)}{f(y)},$$

which is equivalent to the homomorphy condition $\varphi(xy) = \varphi(x)\varphi(y)$. □

Remark. Originally we used the euclidean metric and our result, restated in terms of the arc metric, was equivalent to $\arg|\varphi(x)/f(x)| < 2\varepsilon$ with $\varepsilon \leqslant \pi/22$. For the above improvement, as well for its proof we are indebted to Prof. J.J. Duistermaat. As we have found out later on, a similar idea already occurred in the proofs of stability of the homomorphy equation for mappings from amenable groups and semigroups into the additive group $\mathbb{C}$ by Székelyhidi [17] and Forti [4], respectively,—cf. also Hyers and Rassias [10] and Székelyhidi [18]. As a consequence, the estimates in Sections 4 and 5, compared with the original version, have been considerably improved, as well.

3. GORDON'S THEOREM

In a series of works culminating in [5,6], Gordon developed a nonstandard version of Pontryagin–van Kampen duality for certain quotients of subgroups of hyperfinite abelian groups within an ω_1-saturated nonstandard universe. We will actually need a special case of one of his results, only, formulated in terms of ultraproducts of abelian groups with respect to a nontrivial (hence countably incomplete) ultrafilter over the set $\mathbb{N}$. On the other hand, we will slightly generalize this result from hyperfinite to all internal groups. This could be done just by an inspection of

Gordon's proof. However, to keep the paper (almost) self-contained and, at the same time, accessible to readers not acquainted with nonstandard analysis, we give the proof of that way generalized ultraproduct version of Gordon's theorem in which the nonstandard framework is reduced just to the ω_1-saturation of the ultraproduct.

For Pontryagin–van Kampen duality the reader is referred to [13] or [15] and for the ultraproduct construction, Łoś theorem and transfer principle to [3] and [9].

Let D be a nontrivial ultrafilter on the set $\mathbb{N}$. Given a sequence of groups $(G_i)_{i\in\mathbb{N}}$, let us denote by

$$G = \prod_{i\in\mathbb{N}} G_i/D$$

the ultraproduct of the system (G_i) with respect to the ultrafilter D. Thus G is the quotient of the direct product $\prod_i G_i$ with respect to the normal subgroup

$$\left\{(x_i) \in \prod_i G_i;\ \{i \in \mathbb{N};\ x_i = 1\} \in D\right\}.$$

The coset of the sequence (x_i) in this quotient will be denoted by $(x_i)/D$.

Each sequence $S_i \subseteq G_i$ of subsets of the groups G_i gives rise to a subset $S = (S_i)/D$ of the group G, such that

$$(x_i)/D \in S \quad \Leftrightarrow \quad \{i \in \mathbb{N};\ x_i \in S_i\} \in D,$$

for any $(x_i) \in \prod_i G_i$. Thus $(S_i)/D$ can naturally be identified with the ultraproduct $\prod_i S_i/D$. Subsets of G of this form are called *internal.*

If $H = \prod_i H_i/D$ is the ultraproduct of another system of groups $(H_i)_{i\in\mathbb{N}}$ with respect to the same ultrafilter D and $f_i : G_i \to H_i$ are arbitrary mappings then the formula

$$f(x) = \bigl(f_i(x_i)\bigr)/D,$$

for $x = (x_i)/D \in G$, defines an *internal mapping* $f = (f_i)/D : G \to H$. By Łoś theorem, an internal mapping $f = (f_i)/D : G \to H$ is a group homomorphism if and only if $\{i \in \mathbb{N};\ f_i \in \operatorname{Hom}(G_i, H_i)\} \in D$. Is it the case, then f is called an *internal homomorphism.*

Let us stress that by far not all subsets $X \subseteq G$ are internal; in particular, no infinite countable subset of G is. Similarly, there exist non-internal mappings and non-internal homomorphisms $G \to H$, in general. On the other hand, every *finite* set $F \subseteq G$ is internal.

The system of all internal subsets of G is featured by the following property, called ω_1-saturation. For the proof the reader is referred, e.g., to [1] or [9].

Lemma 3.1. *Let $G = \prod_{i\in\mathbb{N}} G_i/D$ be an ultraproduct of a system of groups G_i with respect to a nontrivial ultrafilter D on $\mathbb{N}$, and $(S_n)_{n\in\mathbb{N}}$ be a system of nonempty internal subsets of G such that $S_{n+1} \subseteq S_n$ for each n. Then $\bigcap_n S_n \neq \emptyset$.*

Let us denote by ${}^*\mathbb{T} = \mathbb{T}^{\mathbb{N}}/D = \prod_{i\in\mathbb{N}} \mathbb{T}/D$ the ultrapower of the multiplicative group of complex units. Identifying each element $a \in \mathbb{T}$ with the coset $(a)/D$ of the constant sequence $a_i = a$, $\mathbb{T}$ becomes an elementary subgroup of its ultrapower ${}^*\mathbb{T}$. As $\mathbb{T}$ is compact, there is a retraction homomorphism ${}^\circ : {}^*\mathbb{T} \to \mathbb{T}$, called the *standard part map*, given by

$$^\circ(a_i)/D = D\text{-}\lim a_i,$$

where D-lim denotes the limit with respect to the ultrafilter D. For points $a = (a_i)/D$, $b = (b_i)/D$ in ${}^*\mathbb{T}$ we put

$$a \approx b \quad \Leftrightarrow \quad {}^\circ a = {}^\circ b.$$

In that case a and b are said to be *infinitesimally close*. It can easily be seen that the set $\operatorname{mon}(1) = \{a \in {}^*\mathbb{T};\ a \approx 1\}$, called the *monad* of $1 \in {}^*\mathbb{T}$, is a (non-internal) subgroup of ${}^*\mathbb{T}$ and $\mathbb{T} \cong {}^*\mathbb{T}/\operatorname{mon}(1)$.

Theorem 3.2. *Let $G = \prod_{i\in\mathbb{N}} G_i/D$ be an ultraproduct of a system of abelian groups G_i with respect to a nontrivial ultrafilter D on $\mathbb{N}$, and X be a countable subgroup of G. Then for each character $g : X \to \mathbb{T}$ there exists an internal character $\gamma : G \to {}^*\mathbb{T}$ such that*

$$g(x) = {}^\circ\gamma(x),$$

for each $x \in X$.

Proof. Let $\Gamma_i = \widehat{G}_i = \operatorname{Hom}(G_i, \mathbb{T})$ denote the dual group of G_i, and

$$\Gamma = \prod_{i\in\mathbb{N}} \Gamma_i/D.$$

Thus the elements of Γ are exactly all the *internal* characters $\gamma : G \to {}^*\mathbb{T}$, and (neglecting topology) Γ plays the role of the dual group of G within the "world of internal objects". Similarly, $\widehat{X} = \operatorname{Hom}(X, \mathbb{T})$ denotes the dual group of the discrete abelian group X. Thus $\widehat{X}$ is a compact metrizable topological group.

Consider the map $\Phi : \Gamma \to \widehat{X}$ given by $\Phi(\gamma) = {}^\circ\gamma \restriction X$, i.e., $\Phi(\gamma)(x) = {}^\circ\gamma(x)$ for $\gamma \in \Gamma$, $x \in X$. Obviously, Φ is a group homomorphism. The proof will be complete once we show that Φ is onto. To this end it is enough to prove that $\Phi[\Gamma]$ is both closed and dense in $\widehat{X}$.

Let us recall that the canonical neighborhood basis of the unit element in $\widehat{X}$ is formed by the sets

$$\widehat{X}(F, \Lambda_\delta) = \{g \in \widehat{X};\ g[F] \subseteq \Lambda_\delta\},$$

with F running over all finite subsets of X, and $0 < \delta < 2\pi/3$, where

$$\Lambda_\delta = \{c \in \mathbb{T};\ |\arg c| \leqslant \delta\}.$$

Similarly, we put

$$^*\Lambda_\delta = \Lambda_\delta^{\mathbb{N}}/D = \{c \in {}^*\mathbb{T};\ |\arg c| \leqslant \delta\},$$

where, for $c = (c_i)/D \in {}^*\mathbb{T}$, the last inequality means that $\{i \in \mathbb{N};\ |\arg c_i| \leqslant \delta\} \in D$.

Let us fix some strictly decreasing sequence $\delta_n \to 0$, $\delta_0 < 2\pi/3$, and a sequence of finite sets $F_n \subseteq F_{n+1}$ such that $X = \bigcup_n F_n$. Then the sets $\widehat{X}(F_n, \Lambda_{\delta_n})$ form a countable neighborhood basis of the unit element in $\widehat{X}$. Assume that (γ_k) is a sequence in Γ such that the sequence $g_k = {}^\circ\gamma_k \restriction X \in \Phi[\Gamma]$ converges to a $g \in \widehat{X}$. Then for each n there is an $m \in \mathbb{N}$ such that $g_k \in g\widehat{X}(F_n, \Lambda_{\delta_n})$ whenever $k \geqslant m$, i.e., $\gamma_k \in \Phi^{-1}[g\widehat{X}(F_n, \Lambda_{\delta_n})]$, in particular, $\Phi^{-1}[g\widehat{X}(F_n, \Lambda_{\delta_n})] \neq \emptyset$. Similarly one can show that $\Phi^{-1}[g\widehat{X}(F_n, \Lambda_{\delta_{n+1}})] \neq \emptyset$.

Put

$$\begin{aligned} U_n &= \{\gamma \in \Gamma;\ (\forall x \in F_n)(\gamma(x) \in g(x)^*\Lambda_{\delta_n})\} \\ &= \bigcap_{x \in F_n} \{\gamma \in \Gamma;\ \gamma(x) \in g(x)^*\Lambda_{\delta_n}\}. \end{aligned}$$

As the system of all internal sets is closed under finite boolean operations, the equality

$$\{\gamma \in \Gamma;\ \gamma(x) \in g(x)^*\Lambda_\delta\} = \prod_{i \in \mathbb{N}} \Gamma_i(x_i, g(x)\Lambda_\delta)/D,$$

true for any $x = (x_i)/D \in G$, $\delta > 0$, where $\Gamma_i(x_i, A) = \{\gamma_i \in \Gamma_i;\ \gamma_i(x_i) \in A\}$ for $A \subseteq \mathbb{T}$, shows that each of the sets U_n is internal. Let us prove the inclusion

$$\Phi^{-1}[g\widehat{X}(F_n, \Lambda_{\delta_{n+1}})] \subseteq U_n.$$

Any γ from the set on the left satisfies ${}^\circ\gamma(x) \in g(x)\Lambda_{\delta_{n+1}}$, for each $x \in F_n$. Since $\delta_{n+1} < \delta_n$, this implies $\gamma(x) \in g(x)^*\Lambda_{\delta_n}$, i.e., $\gamma \in U_n$.

Thus (U_n) is a sequence of nonempty internal subsets of Γ. Since $U_{n+1} \subseteq U_n$ is obvious, by ω_1-saturation there is a $\gamma \in \bigcap_n U_n$. Then $\gamma(x) \approx g(x)$ for each $x \in X$, hence $g = {}^\circ\gamma \restriction X \in \Phi[\Gamma]$. This shows that $\Phi[\Gamma]$ is closed in $\widehat{X}$.

In order to prove that $\Phi[\Gamma]$ is dense in $\widehat{X}$, it suffices to show that it separates points in X. Take any $x = (x_i)/D \neq 1$ in X and choose a $\delta \in (0, 2\pi/3)$. Then there is a set $J \in D$ such that $x_i \neq 1$ for all $i \in J$. For any $i \in J$ one can find a $\gamma_i \in \Gamma_i$ such that $\gamma_i(x_i) \notin \Lambda_\delta$. Let $\gamma_i : G_i \to \mathbb{T}$ be the trivial character for $i \in \mathbb{N} \setminus J$. Then $\gamma = (\gamma_i)/D \in \Gamma$ and $\gamma(x) \notin {}^*\Lambda_\delta$, hence ${}^\circ\gamma(x) \neq 1$. □

Remark. One can actually prove more than stated in Theorem 3.2. Indeed, the infinitesimal annihilator of X in Γ, i.e.,

$$X^\perp = \Gamma(X, \mathrm{mon}(1)) = \{\gamma \in \Gamma;\ (\forall x \in X)(\gamma(x) \approx 1)\},$$

is a subgroup of Γ, and all the internal sets S satisfying $X^\perp \subseteq S \subseteq \Gamma$ form a neighborhood basis for a non-Hausdorff (unless X is finite), compact group topology on Γ. Then $\Gamma/X^{\perp}$ endowed with the quotient topology already is a Hausdorff, compact topological group. Since, obviously, Φ is continuous and $\operatorname{Ker}\Phi = X^{\perp}$, it induces a continuous 1–1 homomorphism $\Phi' : \Gamma/X^{\perp} \to \widehat{X}$ of topological groups. Φ' is, in fact, a canonic isomorphism of the topological group $\Gamma/X^{\perp}$ onto the dual $\widehat{X}$. Details, as well as some more general results, can be found in [5] and [6].

4. APPROXIMATE EXTENSION OF PARTIAL ε-CHARACTERS TO CHARACTERS

Now, we are ready to prove the first of the couple of results announced at the beginning of our article.

Theorem 4.1. *Let $0 < \delta < \varepsilon \leqslant \pi/2$ and $1 \leqslant q \in \mathbb{N}$. Then there exists a positive integer $n \in \mathbb{N}$ (depending just on δ, ε and q) such that for any abelian group G, a set $S \subseteq G$, satisfying $\#S \leqslant q$, and a partial δ-character $f : \langle S \rangle_n \to \mathbb{T}$ there is a character $\varphi : G \to \mathbb{T}$ such that*

$$\left| \arg \frac{\varphi(x)}{f(x)} \right| < \varepsilon,$$

for each $x \in S$.

Proof. Assume the contrary and choose a δ, ε and q witnessing it. Then there is a sequence G_i of abelian groups with subsets $S_i \subseteq G_i$, $\#S_i \leqslant q$, and partial δ-characters $f_i : \langle S_i \rangle_i \to \mathbb{T}$, such that for each genuine character $\varphi_i : G_i \to \mathbb{T}$ there is an $x_i \in S_i$ subject to

$$\left| \arg \frac{\varphi_i(x_i)}{f_i(x_i)} \right| \geqslant \varepsilon.$$

Let D be any nontrivial ultrafilter on the set $\mathbb{N}$ and $G = \prod_{i \in \mathbb{N}} G_i / D$. Put

$$S_{ik} = \langle S_i \rangle_k$$

for $i, k \in \mathbb{N}$. Then $S_{ik} \subseteq G_i$ and $S_{ik} S_{il} \subseteq S_{ik+l}$ for any i, k, l. Moreover

$$\#S_{ik} \leqslant d_{qk} = \#\{(a_1, \ldots, a_q) \in \mathbb{Z}^q;\ |a_1| + \cdots + |a_q| \leqslant k\},$$

which is a (finite) positive integer depending just on q and k. Let us denote

$$X_k = \prod_{i \in \mathbb{N}} S_{ik}/D \subseteq G \quad \text{and} \quad X = \bigcup_{k \in \mathbb{N}} X_k.$$

By transfer principle, X_k are symmetric subsets of G containing 1, $X_k X_l \subseteq X_{k+l}$, and $\#X_k \leqslant d_{qk}$, as well. Thus X is a countable subgroup of G.

The internal map $f = (f_i)/D$ is a partial δ-homomorphism with domain $S = \prod_i S_{ii}/D$. For any $k \in \mathbb{N}$ we have $S_{ik} \subseteq S_{ii}$ whenever $k \leqslant i$, hence $X_k \subseteq S$ as well as $X \subseteq S$. Thus ${}^\circ f \restriction X : X \to \mathbb{T}$ is a δ-homomorphism. As the discrete abelian group

X is amenable, by Theorem 2.1 there is a homomorphism $g : X \to \mathbb{T}$ such that

$$\left|\arg \frac{g(x)}{{}^\circ f(x)}\right| \leqslant \delta < \varepsilon$$

for $x \in X$. By Gordon's Theorem 3.2, there is an internal character $\gamma = (\gamma_i)/D : G \to {}^*\mathbb{T}$ such that $g(x) = {}^\circ\gamma(x)$ for each $x \in X$. Hence

$$\left|\arg \frac{\gamma(x)}{f(x)}\right| < \varepsilon.$$

On the other hand, there is a set $J \in D$ such that $\gamma_i : G_i \to \mathbb{T}$ is a character for each $i \in J$. By the original assumption, there is an $x_i \in S_i$ such that

$$\left|\arg \frac{\gamma_i(x_i)}{f_i(x_i)}\right| \geqslant \varepsilon.$$

Put $x_i = 1$ for $i \in \mathbb{N} \setminus J$. Then $x = (x_i)/D \in X_1 \subseteq X$ and

$$J \subseteq \big\{i \in \mathbb{N};\ \big|\arg\big(\gamma_i(x_i)/f_i(x_i)\big)\big| \geqslant \varepsilon\big\}.$$

Hence the last set is in D, too. Therefore,

$$\left|\arg \frac{\gamma(x)}{f(x)}\right| \geqslant \varepsilon.$$

This contradiction proves the theorem. □

We denote by $\nu(\delta, \varepsilon, q)$ the first integer $n \geqslant 1$ satisfying the condition of Theorem 4.1 for δ, ε and q.

5. APPLICATION TO DUALS OF INTEGRAL POINT LATTICES

For any fixed $q \in \mathbb{N}$, the dual group $\widehat{\mathbb{Z}^q} = \mathrm{Hom}(\mathbb{Z}^q, \mathbb{T})$ of $\mathbb{Z}^q$ is canonically isomorphic to $\mathbb{T}^q$; the action of an $\alpha = (\alpha_1, \dots, \alpha_q) \in \mathbb{T}^q$ on $\mathbb{Z}^q$ is given by

$$x \mapsto \alpha^x = \alpha_1^{x_1} \cdots \alpha_q^{x_q},$$

for $x = (x_1, \dots, x_q) \in \mathbb{Z}^q$. For any set $X \subseteq \mathbb{Z}^q$ we denote by

$$X' = \big\{\alpha \in \mathbb{T}^q;\ (\forall x \in X)(\alpha^x = 1)\big\}$$

the annihilator of X in $\mathbb{T}^q$; it is always a subgroup of $\mathbb{T}^q$. If H is a *subgroup* of $\mathbb{Z}^q$, i.e., an integral point lattice in $\mathbb{R}^q$, then, by the Pontryagin–van Kampen duality, there are canonic group isomorphisms $\widehat{H} \cong \mathbb{T}^q/H'$ and $\widehat{\mathbb{Z}^q/H} \cong H'$.

We also denote

$$B_1 = \big\{x \in \mathbb{R}^q;\ \|x\|_1 \leqslant 1\big\} \quad \text{and} \quad B_\infty = \big\{x \in \mathbb{R}^q;\ \|x\|_\infty \leqslant 1\big\},$$

the closed unit balls with respect to the ℓ_1-norm $\|x\|_1 = \sum_{i=1}^q |x_i|$, and with respect to the ℓ_∞-norm $\|x\|_\infty = \max_{i \leqslant q} |x_i|$ on $\mathbb{R}^q$, respectively. The interior of any set

$X \subseteq \mathbb{R}^q$ is denoted by X°. Similarly,

$$\Lambda_\delta^\circ = \{c \in \mathbb{T};\ |\arg c| < \delta\}$$

denotes the interior of the arc Λ_δ in $\mathbb{T}$, and

$$\mathbb{T}^q(X, A) = \{\alpha \in \mathbb{T}^q;\ (\forall x \in X)(\alpha^x \in A)\},$$

for any $X \subseteq \mathbb{Z}^q$, $A \subseteq \mathbb{T}$.

Theorem 5.1. *Let $0 < \delta < \varepsilon \leqslant \pi/3$ and $1 \leqslant q \in \mathbb{N}$. Then there exists a positive integer $m \in \mathbb{N}$ (depending just on δ, ε and q) such that for every integral point lattice $H \leqslant \mathbb{Z}^q$ we have*

$$\mathbb{T}^q(H \cap mB_1, \Lambda_\delta) \subseteq H' \cdot (\Lambda_{2\varepsilon}^\circ)^q,$$

i.e., for each $\alpha \in \mathbb{T}^q$, satisfying $\alpha^x \in \Lambda_\delta$ for any $x \in H \cap mB_1$, there is a $\beta \in H'$, such that $|\arg(\alpha_j/\beta_j)| < 2\varepsilon$ whenever $1 \leqslant j \leqslant q$.

Proof. Let $G = \mathbb{Z}^q/H$, $s_j = e_j/H \in G$, where e_j is the unit vector in $\mathbb{Z}^q$ with 1 in the jth place and 0's elsewhere, and $s = (s_1, \dots, s_q)$. Then G is an abelian group generated by the set $S = \{s_1, \dots, s_q\}$, $\#S \leqslant q$, and the canonic projection $\eta: \mathbb{Z}^q \to G$ sending e_j to s_j is given by $\eta(x) = xs = x_1 s_1 + \cdots + x_q s_q$, for $x = (x_1, \dots, x_q) \in \mathbb{Z}^q$.

For any $n \in \mathbb{N}$ we have

$$\langle S\rangle_n = \{xs;\ x \in \mathbb{Z}^q \cap nB_1\}.$$

Keep n fixed and put

$$P_a = \eta^{-1}[a] \cap 2nB_1 = \{x \in \mathbb{Z}^q \cap 2nB_1;\ xs = a\},$$

for each $a \in \langle S\rangle_n$. If $x, y \in P_a$, then obviously $x - y \in H \cap 4nB_1$.

Let $\alpha = (\alpha_1, \dots, \alpha_q) \in \mathbb{T}^q$ be such that $\alpha^x = \alpha_1^{x_1} \cdots \alpha_q^{x_q} \in \Lambda_\delta$, for any $x = (x_1, \dots, x_q) \in H \cap 4nB_1$; in other words, $\alpha \in \mathbb{T}^q(H \cap 4nB_1, \Lambda_\delta)$. Then $\alpha^{x-y} \in \Lambda_\delta$, whenever $x, y \in P_a$ for some $a \in \langle S\rangle_n$. Since $\delta < \pi/3 < 2\pi/3$, the set $\{\alpha^x;\ x \in P_a\}$ is contained within the arc $\{c \in \mathbb{T};\ |\arg(c/c_a)| \leqslant \delta/2\}$ for some $c_a \in \mathbb{T}$. Any such a choice $a \mapsto c_a$ gives rise to a function $f: \langle S\rangle_n \to \mathbb{T}$ such that

$$\left|\arg\left(f(a)\alpha^{-x}\right)\right| \leqslant \frac{1}{2}\delta,$$

for any $a \in \langle S\rangle_n$, $x \in P_a$. In particular, $|\arg(f(s_j)/\alpha_j)| \leqslant \delta/2$, for $j \leqslant q$. We will show that f is a partial $\frac{3}{2}\delta$-character (with respect to the arc metric on $\mathbb{T}$).

Take any $a, b \in \langle S\rangle_n$, such that $a+b \in \langle S\rangle_n$. Then $a = xs$, $b = ys$, $a+b = (x+y)s$ for some $x \in P_a \cap nB_1$, $y \in P_b \cap nB_1$ and $x + y \in 2nB_1$. Therefore, $x + y \in P_{a+b}$, and

$$f(a)f(b)f(a+b)^{-1} = f(a)\alpha^{-x} f(b)\alpha^{-y}\alpha^{x+y} f(a+b)^{-1}.$$

Hence,

$$\left|\arg\left(f(a)f(b)/f(a+b)\right)\right| \leqslant \frac{3}{2}\delta.$$

If $3\delta/2 < 3\varepsilon/2 \leqslant \pi/2$, i.e., $\delta < \varepsilon \leqslant \pi/3$, and $n = \nu(3\delta/2, 3\varepsilon/2, q)$ then, by Theorem 4.1, there is a character $\varphi : G \to \mathbb{T}$ such that $|\arg(f(s_j)/\varphi(s_j))| < 3\varepsilon/2$ for any $j \leqslant q$. Putting $\beta_j = \varphi(s_j)$ we obtain a $\beta = (\beta_1, \dots, \beta_q) \in H'$ such that $|\arg(\alpha_j/\beta_j)| < 2\varepsilon$ for each j. Thus it suffices to put $m = 4n$. $\square$

We denote by $\mu(\delta, \varepsilon, q)$ the first integer $m \geqslant 1$ satisfying the condition of Theorem 5.1 for δ, ε and q. As it follows from the proof, for any admitted parameters δ, ε and q we have the estimation

$$\mu(\delta, \varepsilon, q) \leqslant 4\nu\left(\frac{3}{2}\delta, \frac{3}{2}\varepsilon, q\right).$$

The last theorem can be strengthened in the following way: allowing m to grow, ε can be made arbitrarily small. Before giving the precise formulation, let us record a technical lemma. Recall that a *body* in $\mathbb{R}^q$ is a nonempty bounded connected set $B \subseteq \mathbb{R}^q$ which is the closure of its interior. Each symmetric convex body $B \subseteq \mathbb{R}^q$ determines the norm $\|x\|_B = \inf\{\lambda \in \mathbb{R};\ \lambda \geqslant 0,\ x \in \lambda B\}$ on $\mathbb{R}^q$; then $B = \{x \in \mathbb{R}^q;\ \|x\|_B \leqslant 1\}$ coincides with the closed unit ball with respect to this norm (see, e.g., [12]).

Lemma 5.2. *Let $0 < \delta < 2\pi/3$ and q, k be positive integers. Then for every integral point lattice $H \leqslant \mathbb{Z}^q$ and any symmetric convex body $B \subseteq \mathbb{R}^q$ we have*

$$\mathbb{T}^q(H \cap kB, \Lambda_\delta) \subseteq \mathbb{T}^q(H \cap B, \Lambda_{\delta/k}).$$

Proof. Let $\alpha \in \mathbb{T}^q(H \cap kB, \Lambda_\delta)$ and $x \in H \cap B$. Denote $c = \alpha^x$. Then $x, 2x, \dots, kx \in H \cap kB$, hence $c, c^2, \dots, c^k \in \Lambda_\delta$. As $\delta < 2\pi/3$, this is possible only if $c \in \Lambda_{\delta/k}$. $\square$

Theorem 5.3. *Let $0 < \delta < 2\pi/3$, $\varepsilon > 0$ and $1 \leqslant q \in \mathbb{N}$. Then there exists a positive integer $n \in \mathbb{N}$ (depending just on δ, ε and q) such that for every integral point lattice $H \leqslant \mathbb{Z}^q$ we have*

$$\mathbb{T}^q(H \cap nB_1, \Lambda_\delta) \subseteq H' \cdot (\Lambda_\varepsilon^\circ)^q,$$

i.e., for each $\alpha \in \mathbb{T}^q$, satisfying $\alpha^x \in \Lambda_\delta$ for any $x \in H \cap nB_1$, there is a $\beta \in H'$, such that $|\arg(\alpha_j/\beta_j)| < \varepsilon$ whenever $1 \leqslant j \leqslant q$.

Proof. Without loss of generality we can assume that $\varepsilon \leqslant 2\pi/3$. Let k be the smallest positive integer such that $\delta/k < \varepsilon/2 \leqslant \pi/3$. Put $m = \mu(\delta/k, \varepsilon/2, q)$ and $n = km$.

Let $\alpha \in \mathbb{T}^q(H \cap nB_1, \Lambda_\delta)$. Then $B = mB_1$ is a symmetric convex body in $\mathbb{R}^q$ and $kB = nB_1$, hence $\alpha \in \mathbb{T}^q(H \cap mB_1, \Lambda_{\delta/k})$ by Lemma 5.2. By Theorem 5.1 and the choice of k and m, there is a $\beta \in H'$ such that $|\arg(\alpha_j/\beta_j)| < \varepsilon$ for $1 \leqslant j \leqslant q$. $\square$

The first integer $n \geqslant 1$ satisfying the condition of Theorem 5.3 for δ, ε and q will be denoted by $N(\delta, \varepsilon, q)$.

Remark. If n were allowed to depend not just on δ, ε and q but also on H, then Theorem 5.3, weakened in this way, would be an almost direct consequence of the Pontryagin–van Kampen duality. One could even prove the following more general result (the just mentioned weakening of Theorem 5.3 would be then a special case for the discrete group $G = \mathbb{Z}^q$ and $S_n = \mathbb{Z}^q \cap nB_1$):

Let G be a σ-compact LCA group and $\widehat{G}$ be its dual group. Assume that $S_n \subseteq S_{n+1}$ is a sequence of symmetric compact neighborhoods of the unit element $1 \in G$, such that $G = \bigcup_{n \in \mathbb{N}} S_n$, and H is a closed subgroup of G. Then for any $\delta \in (0, 2\pi/3)$, $\varepsilon > 0$, there is a positive integer $n \in \mathbb{N}$ such that

$$\widehat{G}(H \cap S_n, A_\delta) \subseteq H' \cdot \widehat{G}(S_1, A_\varepsilon^\circ),$$

where $\widehat{G}(S, A) = \{\varphi \in \widehat{G};\ \varphi[S] \subseteq A\}$, for $S \subseteq G$, $A \subseteq \mathbb{T}$, and $H' = \widehat{G}(H, 1)$ is the annihilator of H in $\widehat{G}$.

Indeed, from the Pontryagin–van Kampen duality we have the canonic isomorphism of topological groups $\widehat{G}/H' \cong \widehat{H}$, induced by the restriction map $\widehat{G} \to \widehat{H}$, $\alpha \mapsto \alpha \restriction H$, which is a surjective continuous homomorphism with kernel H'. For any fixed $\delta \in (0, 2\pi/3)$, the sets

$$\widehat{H}(H \cap S_n, A_\delta) = \big\{\alpha \restriction H;\ \alpha \in \widehat{G}(H \cap S_n, A_\delta)\big\}$$

form a neighborhood basis of the unit element $1 \in \widehat{H}$. The set $\widehat{G}(S_1, A_\varepsilon^\circ)/H'$ is a neighborhood of $1 \in \widehat{G}/H'$. Thus there is an n such that the preimage of the neighborhood $\widehat{H}(H \cap S_n, A_\delta)$ under the canonic isomorphism $\widehat{G}/H' \cong \widehat{H}$ is a subset of $\widehat{G}(S_1, A_\varepsilon^\circ)/H'$. Lifting these sets up to $\widehat{G}$ along the restriction map $\widehat{G} \to \widehat{H}$ and the canonic projection $\widehat{G} \to \widehat{G}/H'$, respectively, we get the desired inclusion.

For any set $X \subseteq \mathbb{R}^q$, let us denote

$$X^+ = \big\{y \in \mathbb{R}^q;\ (\forall x \in X)(xy \in \mathbb{Z})\big\}$$

its integral annihilator, where $xy = x_1y_1 + \cdots + x_qy_q$ is the usual scalar product in $\mathbb{R}^q$. If H is a subgroup of $\mathbb{Z}^q$, i.e., an integral point lattice in $\mathbb{R}^q$, then $\mathbb{Z}^q = (\mathbb{Z}^q)^+ \subseteq H^+$, and the dual group of the quotient $\mathbb{Z}^q/H$ is isomorphic to $H' \cong H^+/\mathbb{Z}^q$. With this in mind, changing the scale from 2π to 1, and introducing the notation

$$X^{(\delta)} = \big\{y \in \mathbb{R}^q;\ (\forall x \in X)(\exists c \in \mathbb{Z})\big(|xy - c| \leqslant \delta\big)\big\},$$

for $X \subseteq \mathbb{R}^q$, $\delta > 0$, one can readily translate Theorem 5.3 into the language of integral annihilators of integral point lattices.

Corollary 5.4. *Let $0 < \delta < 1/3$, $\varepsilon > 0$ and $1 \leqslant q \in \mathbb{N}$. Then for each $n \geqslant N(\delta, \varepsilon, q)$ and every integral point lattice $H \leqslant \mathbb{Z}^q$ we have*

$$(H \cap nB_1)^{(\delta)} \subseteq H^+ + \varepsilon B_\infty^\circ,$$

i.e., for each $u \in (H \cap nB_1)^{(\delta)}$ there is a $v \in H^+$ such that $\|u - v\|_\infty < \varepsilon$.

If $[H]$ denotes the linear span of a point lattice H in $\mathbb{R}^q$, then

$$H^\star = H^+ \cap [H]$$

is the dual (or polar) lattice of H (see, e.g., [12]). If $[H] = \mathbb{R}^q$, i.e., if H is a full rank point lattice in $\mathbb{R}^q$, then, of course, $H^\star = H^+$. If additionally $H \subseteq \mathbb{Z}^q$, then Corollary 5.4, with the same lower bound for n, is true when replacing H^+ by the dual lattice $H^\star$.

Corollary 5.5. *Let $0 < \delta < 1/3$, $\varepsilon > 0$ and $1 \leqslant q \in \mathbb{N}$. Then for each $n \geqslant N(\delta, \varepsilon, q)$ and every full rank integral point lattice $H \leqslant \mathbb{Z}^q$ we have*

$$(H \cap nB_1)^{(\delta)} \subseteq H^\star + \varepsilon B_\infty^\circ,$$

i.e., for each $u \in (H \cap nB_1)^{(\delta)}$ there is a $v \in H^\star$ such that $\|u - v\|_\infty < \varepsilon$.

Moreover, realizing that for any symmetric convex bodies $K_1, K_2 \subseteq \mathbb{R}^q$ there are constants $\lambda_1, \lambda_2 > 0$ (depending just on K_1, K_2, respectively) such that $B_1 \subseteq \lambda_1 K_1$ and $B_\infty \subseteq \lambda_2 K_2$, we get the following result.

Corollary 5.6. *Let $0 < \delta < 1/3$, $\varepsilon > 0$ and $1 \leqslant q \in \mathbb{N}$. Then for any symmetric convex bodies $K_1, K_2 \subseteq \mathbb{R}^q$ there exists a positive integer $m \in \mathbb{N}$ (depending just on δ, ε, q and K_1, K_2) such that for every integral point lattice $H \subseteq \mathbb{Z}^q$ we have*

$$(H \cap mK_1)^{(\delta)} \subseteq H^+ + \varepsilon K_2^\circ,$$

i.e., for each $u \in (H \cap mK_1)^{(\delta)}$ there is a $v \in H^+$ such that $\|u - v\|_{K_2} < \varepsilon$.

Proof. Indeed, with $n = N(\delta, \varepsilon/\lambda_2, q)$ we have

$$(H \cap n\lambda_1 K_1)^{(\delta)} \subseteq (H \cap nB_1)^{(\delta)} \subseteq H^+ + \frac{\varepsilon}{\lambda_2} B_\infty^\circ \subseteq H^+ + \varepsilon K_2^\circ.$$

Thus it suffices to take any $m \geqslant n\lambda_1$. $\square$

In the full rank case, one can replace the integral annihilator H^+ by the dual lattice $H^\star$, again.

Final remark. It would be interesting if somebody could prove either Theorem 4.1 or any of the results of Section 5 in a more constructive way, avoiding the assumption of existence of nontrivial ultrafilters on $\mathbb{N}$, as well as any higher choice related axioms of set theory. E.g., one could try to prove either Theorem 4.1 or Theorem 5.1 by induction on q. Furthermore, restricting the values of δ and ε to some sequences of the form $\delta_k = 1/a_k$, $\varepsilon_k = 1/b_k$, where $a_k > 3$, $b_k > 0$ are some fixed strictly increasing primitive recursive sequences of integers, Corollary 5.5 can relatively easily be re-formulated within the language of Peano arithmetic (PA). Thus it is natural to ask the following question: Is (this modification of) Corollary 5.5 provable in PA? A closely related question can be stated as follows: Let $W(k)$ denote the first $n \geqslant 1$ satisfying the conclusion of Corollary 5.5 for, say, $\delta = 1/(k+4)$, $\varepsilon = 1/(k+1)$, $q = k+1$. Is the function $W: \mathbb{N} \to \mathbb{N}$ (primitive) recursive?

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(Received July 2004)